Infinity is, like, really big. But it's not too big to understand. Philosophers and mathematicians invented set theory to help us all get our heads around infinity. Let's do it!

**Representing sets**

A set of individuals b and c may be represented as either \{b, c\} or \{c, b\} (order doesn’t matter).

Sets can have sets as their members. For instance \{a, \{b,c\}\}.

Sets are distinct from their members. So, for example, a ≠ \{a\}.

A set of all and only individuals satisfying some condition Fx may be symbolized {x: Fx} which may be read as “The set of all x such that x is F”. So, for example, {x: x is a student in Mandik’s class} = the set of all students in Mandik’s class = The set of all x such that x is a student in Mandik’s class.

We can name sets for convenience. For instance, let A = \{a, b, c\} and M = \{x: x is a student in Mandik’s class\}. If M has more than three members, then M ≠ A.

One set has no members at all. It is the empty set. Some ways to represent it are \{} and ∅.
Relations to sets

MEMBERSHIP: If $d$ is a member of the set $H$, we can write that as “$d \in H$”. If $d$ is not a member of $H$, that can be written as “$\neg (d \in H)$” or “$d \not\in H$”. We’ve already seen that sets can be members of sets, so if $A = \{a, \{b, c\}\}$ and $B = \{b, c\}$ then $B \in A$. Note, however, that since sets are distinct from their members, $b \not\in A$ even though $b \in B$ and $B \in A$. Thus is the membership relation an intransitive relation.

SUBSETS and PROPER SUBSETS: $B$ is a subset of $C$ if and only if all members of $B$ are members of $C$. We may write “$B$ is a subset of $C$” as “$B \subseteq C$”. So, for example $\{m, n\} \subseteq \{m, n, o, p\}$. Note also that sets can be subsets of themselves, so $\{m, n\} \subseteq \{m, n\}$.

When a set is a subset of another set without being identical to it, that is, when $B \subseteq C \land B \neq C$, then $B$ is a proper subset of $C$. “$B$ is a proper subset of $C$” may be written as “$B \subset C$”. It is useful to think of the symbolism of “$\subset$” and “$\subseteq$” as related to the symbolism for “less than” and “less than or equal to”.

Ok, now we’re ready to tackle infinity! w00t!!

Infinity

ONE-TO-ONE MAPPING: If two sets have exactly the same number of members, then there is a one-to-one mapping between the two sets. Such sets are equinumerous.

NATURAL NUMBERS: The plain ol’ whole numbers used for counting. The non-negative integers, $\{0, 1, 2, 3,\ldots\}$. There’s controversy about whether to count zero among the naturals. There’s controversy about everything, even in math! It’s a crazy crazy world.

REAL NUMBERS: Any value in the continuum, including positive and negative integers, fractions, and irrationals like PI and the square root of 2.

INFINITE SETS AND FINITE SETS: The notion of an infinite set is defined negatively: any set is infinite if it is not finite. The notion of a finite set is defined like this: A finite set has members that map one-to-one onto the set of all natural numbers less than some natural number $n$. No finite set is equinumerous to any of its proper subsets.

Example: $A = \{a, b, c\}$. There is a one-to-one mapping of the members of $A$ onto the set of all natural numbers less than 3: $a$ maps onto 0, $b$ maps onto 1, and $c$ maps onto 2.
maps onto 2. Note that A has 3 members. None of A's proper subsets are equinumerous to A.

What’s distinctive of infinite sets is that they can have proper subsets that they are equinumerous to. Yup! For instance, the set of all whole numbers and its proper subset, the set of all even whole numbers, are equinumerous. If you aren’t cool with that, then you aren’t cool with infinity.

There’s more!

Some infinities are bigger than others. And, no, you don’t get them by infinity plus one. Nor do you get them by adding infinity to infinity. All of those operations just get you the same old baby-sized infinity. You get the bigger infinities by showing that there’s an infinite set that is not equinumerous to (one-to-one mappable to) the natural numbers. Brace yourself for Georg Cantor, and his trippy diagonalization argument.

CANTOR’S DIAGONALIZATION ARGUMENT:

An infinite set that has members that can be mapped one-to-one onto the natural numbers is a countable set. Georg Cantor proved the existence of infinite sets larger than this. Such sets are uncountable. Cantor proved that there are more real numbers than integers.

If for each of the natural numbers we had a row upon which we wrote the infinite decimal expansion of a real number, then we could discover a number that is not on any of those rows by taking the diagonal (the number whose first digit is the first digit of the first row, second digit is the second digit of the second row, and so on) and changing each of the diagonal’s digits. The resultant changed diagonal is guaranteed not to have been anywhere in the original list since any number on the original list would differ from the changed diagonal at the digit where that number’s row intersected the diagonal.

In the following example the digits in bold italic form the diagonal.
Row 0: 0.1234567…
Row 1: 0.2468101…
Row 2: 0.481632…
Row 3: 0.5101520…
Row 4: 0.98989898…
Row 5: 0.7575757575…

Thus the diagonal number is 0.28087… The changed diagonal is a number that differs from Row 0 in having a different 1st digit, differs from Row 1 in having a different 2nd digit, differs from Row 2 in having a different 3rd digit, and so on. The
changed diagonal will differ from every number on a Row by one digit. Thus, the changed diagonal will not itself be one of the Row numbers. Thus the changed diagonal will not be mapped onto one of the natural numbers. AND... there thus exists at least one more real number (the changed diagonal) than there are natural numbers. Ta da!

A book that I really like that treats infinity in a fun way is *Everything and More: A Compact History Of Infinity* by David Foster Wallace. Wallace was a super-genius novelist with a serious background in philosophy and math. He’s most famous for his novel, *Infinite Jest*.

**Practice problems**

1. [TRUE] [FALSE]: \{a, b\} and \{c, d\} are equinumerous.
2. [TRUE] [FALSE]: \{a, b\} and \{c, d, \emptyset\} are equinumerous.
3. [TRUE] [FALSE]: A finite set may be equinumerous to one of its proper subsets.
4. [TRUE] [FALSE]: An infinite set may be equinumerous to one of its proper subsets.
5. [TRUE] [FALSE]: \{0, 1, 2, 3\} is an infinite set.
6. [TRUE] [FALSE]: \{x: x is a letter of the English alphabet\} is equinumerous to \{0, 1, 2, ... 26\}
7. [TRUE] [FALSE]: The conclusion of Cantor’s diagonalization argument is that there is no set of all sets that are not members of themselves.
8. [TRUE] [FALSE]: The conclusion of Cantor’s diagonalization argument is that there is at least one infinite set that is uncountable.
9. [TRUE] [FALSE]: The set of odd natural numbers is countable.