2 Platonist epistemology and cognition

There can no more be a species of naturalism that is consistent with belief in the existence of abstract objects than there can be a species of atheism that is consistent with belief in the existence of God.
— Jerrold Katz [1990]

So far, I have argued for a very minimal version of mathematical realism. This ‘naked’ ontological realism comprises only two theses:

(R.i) Some mathematical entities exist; and

(R.ii) their existence is not brought about by human activity.

Let’s agree on the strength of the evidence just presented that naked ontological realism (or NORM) is on the right track. As I have emphasized, NORM is merely a rough preliminary sketch of a philosophical position. We are still some distance from a fully developed theory. In this chapter I examine how one influential species of abstract realism—Stewart Shapiro’s [1997] ante rem structuralism—elaborates and extends the basic realist framework.

Although I will focus on one particular account, the point I want to make is general. The vast majority of realists today, including Shapiro himself, hold that mathematical entities (or structures) are abstract and acausal. ‘Realism’ has, in fact, come to be nearly synonymous with ‘platonism’. I think it’s important for philosophers and cognitive scientists to recognize that this is an error. Many of us today share a malaise concerning abstract realism’s apparent inability to explain our knowledge of its posits. In this chapter, I will argue that Shapiro’s own account cannot be accepted as overcoming these difficulties. We shall see, moreover, that some of the difficulties Shapiro encounters are symptomatic of the shortcomings of platonist theories in general. If the acceptance of ante rem posits lands the realist in trouble then, I want to suggest, mathematical realists are well advised to cut their losses, return to NORM and try something new.

2.1 Abstract Realism

Let’s start by taking NORM as our point of departure and retracing the chain of reasoning that leads to platonism.

2.1.1 Platonism

Mathematics studies mathematical objects: sets, functions, groups, numbers, and so on. It characterizes the essential, formal properties of those objects. But what, apart from their naked existence and formal character, can we assert of such entities? On this question,
the mathematician herself remains silent. And so, the first hurdle confronting a philosopher interested in mathematical objects’ nonformal properties concerns methodology: by what means can she determine the answer to this question? One apparently promising way forward is to proceed by conceptual analysis. We can start by drawing up a list of properties (or property types) that mathematicalia cannot exhibit on pain of contradiction or incoherence, thereby arriving at a sort of photonegative of the solution.¹

Here’s an obvious start: Mathematical entities are not observable—at least not in any straightforward sense. They can be represented by means of diagrams, numerals and other such aids. But even so, mathematical proofs are not about the diagrams or the notation; they are about what those represent. Moreover, the unobservability enjoyed by mathematicalia is of a peculiar sort. To see why, let me draw a rough and ready distinction here between two kinds of unobservables. On the one hand, we have entities undetectable by the unaided sensorium in virtue of sheer size, velocity, intense gravitational pull, or some other physical feature. Among such entities we find viruses, quarks, tectonic plates, black holes, and dark matter. A distinct category comprises entities which cannot be touched or seen in virtue of being complexes realized in (or supervening on) other material entities. Instances here include computer files, immune systems, and language acquisition devices. I am not suggesting that this bifurcation is either exclusive or exhaustive. Still, the point can be made that mathematical entities do not fit well on either side. Unlike the cosmologist or geophysicist, the mathematician does not devise or build complex devices so as to better observe her chosen objects. So the obstacle to observing mathematicalia does not seem to stem from our imperfect sensorium [Brown 1999]. Nor does it seem plausible that mathematical entities could be unobservable in virtue of being complexes instantiated in more simple physical systems. One important reason, noted by Frege [1953], has to do with the cardinality of physical entities. Consider: all physical entities are located in space and time. One standard way to construe space-time is as a set of (at most) $2^{\aleph_0}$ points. There are, moreover, at most, a finite number of physical entities in each region of space-time. If so, then the number of physical entities is bounded. There are, in fact, no more than $2^{\aleph_0}$ of them [Parsons 1975].² Clearly, this is a colossal number. Still, standard set theory (with the axiom of replacement) permits the construction of sets of the cardinality $\kappa_\omega$ — a cardinality orders of magnitude larger than any collection of physical objects could possibly be.³ And the iterative hierarchy climbs higher still. So although it may not be immediately clear to a philosopher what it means for these dizzying collections to ‘exist,’ it cannot mean that they

¹This strategy is mentioned, for instance, by Burgess and Rosen [1997].

²To be on the safe side, I’m assuming here that no more than a countable infinity of distinct entities can occupy each space-time point. This is probably overly generous.

exist in virtue of being instantiated in concrete physical models.\(^4\)

There is, additionally, a second reason not to equate mathematical entities with physical objects. I have already noted that mathematical facts are metaphysically necessary; it could not have been the case that they were otherwise. Try as we might, we cannot imagine coherent possible worlds where (say) there exists a largest prime or where \(\sqrt{2}\) is a rational fraction. By contrast, all physical facts are (arguably) in principle contingent. As far as we know, even very fundamental physical facts, such as the values of physical constants, might have been otherwise. The possibility, in any case, appears coherent even if possible worlds where the values are very different from the actual would be uninhabitable by us. The necessity of mathematical facts and contingency of physical ones makes it implausible that the former can be equated with the latter.\(^5\)

Continuing with our conceptual analysis, we find another clue to the properties of mathematical entities in ordinary linguistic usage. To ask where \(\omega\)-sequences are located or for how long the conic sections have existed is to pose nonsensical questions. The problem is not that we are currently ignorant of the answers. Rather, it’s hard to make sufficient sense of what is being asked to know how to go about formulating a reply. These questions commit what Ryle [1949] called a category mistake; they attempt to apply a predicate to a subject matter which is inherently unsuitable to it.

Taken together, these considerations permit us now to venture a first, tentative step beyond naked realism. The position arrived at, modern platonism, is characterized by the acceptance of two hypotheses over and above what \textsc{norm} already commits us to. The first is this:

\((P.i)\) Mathematicalia are imperceptible, atemporal, and nonspatial.

The second hypothesis follows immediately from the first. As far as we know, causal interactions involve entities (roughly) localizable in space-time. But since mathematicalia seem neither to be spatial nor temporal, it follows that they are incapable of playing a role in causal interactions. They cannot be generated, do not decay, and cannot be destroyed. Needless to say, this makes them highly unusual (and perhaps even unique) objects.

\((P.ii)\) Mathematicalia are incapable of entering into causal interactions.

Modern platonism is sometimes called an ante rem realism since the platonist holds that the existence of mathematicalia is independent of and metaphysically prior to the existence

\(^4\)This leaves open the question of whether mathematical entities can be identified with possible physical objects. I defer discussion of modal construals of mathematics for another time.

\(^5\)To what extent conceivability is a guide to possibility is a question to which I will return.
of extended substance (*res extensa*). Next, I’d like to take a closer look at what is perhaps the most sophisticated current version of *ante rem* realism: Stewart Shapiro’s 1997 structuralism.

### 2.1.2 Structuralism

Stewart Shapiro [1997] endorses both (P.i) and (P.ii). But it would be misleading to call his *ante rem* structuralism a species of platonism without further comment. The details of the structuralist conception of mathematics won’t play a significant role in our argument until Chapter 5. Nonetheless, it’s only fair to briefly clarify where the differences between standard platonism and abstract structuralism lie.

Consider for a moment the sorts of entities that, according to the platonist, populate the world of mathematics. We find the mathematical realm teeming with groups, numbers, graphs, functions, sets, classes and other, more exotic species. It’s worth enquiring whether all of these entities are *sui generis* or whether some are ontologically more basic than others. As is well known, the mathematical realist can make significant economies in her basic ontology by supposing that, in the final count, almost all denizens of the platonic realm reduce to sets. That’s because (remarkably enough) we can construct surrogates for just about all mathematical objects by using only the primitive notions of membership and the null set. This does not, of course, imply that all mathematics is just set theory any more than the ontologically basic status of subatomic particles entails that all physical science is just particle physics. The various branches of mathematics have their distinctive intellectual styles, techniques, and problems.

Still, by recognizing the logical priority of sets we gain a natural way of organizing our ontology.

Taking set theory as ontologically fundamental raises certain problems. It’s reasonable to suppose, Benacerraf [1965] argues, that *bona fide* entities have stable identity conditions. Suppose we know, for example, that a certain amino acid is really a particular organic molecule. The amino acid’s various properties (molecular mass, polarity, acidity or basicity and so on) can be accounted for directly in terms of its underlying chemical structure. Moreover, armed with our chemical analysis, we can sensibly ask whether (for instance) the amino acid in question contains or fails to contain a sulphur group (or whatever). In the

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6It’s useful to notice that platonism is a form of ontological dualism. Note also that some of the same argumentative strategies that are normally deployed in favour of mind-body dualism—in particular, arguments from conceivability and from category mistakes—play a role in sustaining platonism.

7Set theory is not our only choice here; one can take mappings as basic and avail oneself of category theory instead. Cf. Hellman [2003]. I return to this topic in later chapters.

8The importance of resisting greedy reductionism in mathematics is stressed by Maddy [1997].

9Some would argue that we can also make advances in epistemology. If we could explain our knowledge of sets then we could thereby explain our knowledge of other mathematicalia. This strategy is pursued, for example, by Maddy [1990].
case of the proposed identification of complex mathematical entities with sets, things are not nearly as clean. Consider the natural numbers. If numbers are objects, we should be able to specify precisely what kind of objects they are. If they are *au fond* sets then we should be able to say exactly *which* sets. As is well known however, there is an infinite variety of non-equivalent ways of constructing a bijective mapping from sets to the natural numbers. Because of this, apparently straightforward factual questions—such as whether or not $1 \in 3$—cannot be settled except by fiat. (If we identify the positive integers with von Neumann’s ordinals then 1 is indeed an element of 3; if we adopt Zermelo’s characterization instead, it’s not.) Moreover, the problem ramifies. There are many nonequivalent ways of using sets to offer surrogates for the integers, rationals, reals, and so on. But if that’s right, Benacerraf argues, then this speaks against identifying numbers, or mathematical entities in general, with sets. In fact it speaks against the notion that mathematicalia are *bona fide* entities at all.

*Ante rem* structuralism offers an elegant reply. The apparent difficulty stems from supposing that mathematical entities are to be thought of as being characterized by their essential, intrinsic properties. There is however an alternative:

Mathematical objects [so numbers, groups, sets] are featureless, abstract positions in structures (or more suggestively, patterns):... paradigm mathematical objects are geometric points, whose identities are fixed only through their relationships to each other.  

*Resnik 1997*

On the structuralist account, rather than investigating discrete entities with (as it were) a mysterious inner nature, mathematics studies positions in abstract patterns. The nature of a mathematical object is fully determined by the place it occupies in such a pattern—which is to say, by its external relations to the structure’s other positions. The picturesque metaphor can be given precise content. *Shapiro 1997* offers axioms that detail the nature of abstract structures. These closely parallel second-order ZF axioms, thus ensuring that the structuralist’s proposal is sufficiently rich to offer a background ontology for the whole of mathematics (on the assumption, of course, that ZF does). The apparent problem of multiple reductions that *Benacerraf 1965* points out emerges as a natural corollary of the theory. Comparisons between elements *within* a single structure (such as questions whether $1 < 3$, and so on) are perfectly sensible and receive answers. But since structuralism only specifies objects ‘up to isomorphism,’ comparisons between objects *across* different structures find no principled solutions. Nor would we expect them to. To be a natural number *just is* to play a role in a structure specified by the (second order, so categorical) PA axioms. It should not surprise us that a variety of distinct sets are able to play this role.

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10Note also that structuralism starts us on the path to explaining why the study of mathematics is so useful in natural science: mathematical patterns can serve as descriptions of concrete, physical systems when the latter happen to display a structure isomorphic to the former.
For our purposes, the important point to take away is that abstract structuralism takes on board both of the propositions that more traditional platonism is committed to. In addition, it accepts a further, logically independent postulate:

(P.iii) Mathematicalia are fully defined by their formal relational properties.

This postulate is logically independent of (P.i) and (P.ii); I return to it in later chapters.

2.2 Knowledge

Earlier, I mentioned the pervasive perception that mathematical realists (of all persuasions) face some tough questions about epistemology. Addressing this topic, Benacerraf [1973] argues as follows: In order to come to know a new fact $F$ several conditions must be satisfied. Most obviously, $F$ must actually hold. As well, since knowledge is a species of belief, coming to know a fact involves either forming a new belief or shifting preexisting and erroneous beliefs with regard to $F$. Finally, to count as knowing that $F$, as opposed to merely having made a lucky guess, we require that appropriate evidentiary grounds exist for our epistemic state. Those grounds are typically construed as the existence of an appropriate connection between ourselves and that which is known. In the case of our knowledge of perceptually observable physical phenomena this connection can plausibly be traced to the causal interaction between our sensory apparatus and our surroundings. Where no connection exists, or in cases where the existing connection fails to be sensitive to the appropriate facts, we cannot be said to have knowledge of $F$.

What troubles Benacerraf about our apparent knowledge of mathematical entities (as construed by platonists and their intellectual successors) is that it’s very hard to say what the relevant epistemic grounds might be. It’s implausible that our knowledge of mathematics beyond grade-school—the sort needed to grasp ZFC, for instance—is innately given. Mathematical research is just too difficult and time-consuming for wholesale nativism to be a plausible hypothesis. Yet since abstract entities (including ante rem structures) do not, ex hypothesi, enter into causal interactions, we cannot easily explain our knowledge of them by analogy with our knowledge of physical facts. Admittedly, it might seem tempting to pass the buck to the mathematician by suggesting that accepted mathematical proofs themselves constitute sufficient evidence for the existence of the relevant objects. Proofs, after all, are considered sufficient evidence for the existence of the relevant mathematical structures by working mathematicians. But, Benacerraf argues, this should not satisfy an ontologist. What is at issue for her is truth and not theoremhood or mathematical correctness. Lacking an independent philosophical account of how mathematical theoremhood manages to track mathematical fact one can sensibly deny that we have reliable mathematical knowledge: the proofs might be correct, but the propositions they demonstrate might still not be true.
Benacerraf’s argument is not a refutation of platonism. The platonist can reject some of the presuppositions that Benacerraf relies on. Among the more vulnerable premises is the causal account of epistemic grounding [Goldman 1967, Skyrms 1967]. Causal accounts promise an attractive way of circumventing Gettier [1963] scenarios. They do however beg the question against the platonist by building in precisely what she explicitly denies—namely, that knowledge requires causal traffic between knower and known. Of course, the accusation of circularity alone does not prove that causal accounts of epistemic grounding are false. For all we know, some version of such account may be exactly right. But causal accounts of epistemic grounding cannot be used to construct compelling arguments against the platonist position. On balance then, Benacerraf [1973] should be read as merely pointing out that no generally acceptable account of our mathematical knowledge yet exists.

The burden of proof, I think, distributes evenly to both sides: It’s incumbent on skeptics about acausal abstracta to show that a non-circular version of Benacerraf’s arguments can be formulated or to furnish an alternative theory which avoids the problematic commitments. The abstract realist, by contrast, needs to make every effort to explain how we might make sense of our knowledge of acausal, abstract facts. Shapiro [1997] takes up this challenge and it is to his account of epistemic contact that I now turn.

2.2.1 Ante rem account

Shapiro [1997] proposes an admirably lucid and detailed epistemic account. On that account, we derive our mathematical knowledge from three sources: pattern recognition, linguistic abstraction, and functional (or implicit) definition. Let me briefly explain each of these routes before critically evaluating the hypothesis being advanced.

Pattern recognition. Of the three sources of mathematical knowledge, pattern recognition is the simplest but also the most limited. Typically, the sensorium of an animate creature is capable of registering and distinguishing a certain range of visual, haptic, olfactory and auditory stimuli. The recurring properties of these stimuli—including, for instance, shape, texture, and pitch—can be registered as well. Sufficiently complex creatures are able to track groups or sequences of recurring properties. Human infants are no exception. For instance, newborns are already capable of distinguishing patterns consisting of two auditory stimuli from those consisting of three [Bijeljac-Babic et al. 1993]. They are capable of distinguishing faces (especially mother’s faces) from other types of patterns and they show a preference for looking at the former [Pascalis et al. 1995]. And there is good evidence that even prior to acquisition of language, infants factor the world in terms of predicates

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11This is sometimes forgotten. See Nozick [1981] for some interesting comments.
12For a discussion of some unexpected limitations of this ability in higher mammals, see Spelke [2002].
and arguments [Gleitman and Fisher 2005]. Finally, in addition to any innate capacities for specific pattern recognition and stimulus parsing, children are able to learn which new types of patterns they need to track; they are, in effect, capable of genuine conceptual advances.\footnote{Shapiro [2000] suggests that “pattern recognition is a deep and challenging problem in cognitive psychology, and [that] there is no accepted account of the underlying mechanisms.” This is not false but it is slightly misleading. Our knowledge of how humans categorize is, of course, incomplete. Nonetheless, standard cognitive science textbooks, including O’Reilly and Munakata [2000] and Gurney [1997], contain fairly sophisticated models and discussions.}

The ante rem structuralist invites us to consider an example of this sort of general learning. Take as our example a child being taught to recognize the letter ‘L’. She may start by learning the alphabet song. Part of memorizing the song involves learning that its twelfth term is a particular voiced, alveolar consonant. Next, the child might be taught to correlate occurrences of that phoneme with written tokens of a specific shape. This is not entirely straightforward since written L-tokens vary considerably in visual appearance. Once she masters that skill, she moves on to still more complex challenges. It turns out that the letter ‘L’ can be tapped in Morse code, gestured in semaphore, and presented as a mariner’s flag. What’s more, certain letter tokens belonging to non-Roman or non-standard alphabets—including λ, Λ, and L—can count under certain conditions as borderline members of the L-type. But in any case, the ante rem realist argues that normal children do eventually come to recognize context-sensitive, multi-modal patterns and that this occurs on the basis of an exposure to concrete tokens. To make sense of this fact the structuralist suggests that we need to admit that an acquaintance with concrete physical tokens can give rise to knowledge of places in a pattern. After all, the only feature shared by the various ‘Ls’ is the role they play in an alphabet. And an alphabet just is a kind of structure. Thus, to make sense of what takes place, we must come to recognize the existence of structures in addition to individual, concrete objects and their properties.

On the ante rem structuralist’s reconstruction, something analogous if somewhat more sophisticated takes place when children come to understand natural numbers. Relatively early on, infants recognize that many otherwise very different physical stimuli—such as light flashes, sequences of tones, and concrete objects—can share a single, higher-order feature: what we would call ‘numerosity’. To recognize this, the infant mustn’t focus unduly on the accidental physical features of each stimulus but rather to attend to multi-modal, context-sensitive information. Adults may perhaps help by actively pointing out various collections while labelling them with the appropriate number term, but in all likelihood

![Figure 1: Tokens of a single type.](image)
the underlying ability is innate \cite{Gallistel2005}. The important point is that learning to attend to and distinguish collections of various numerosities is analogous (the argument runs) to grasping the alphabet. In the one case, the child learns that certain tokens count as ‘Ls’; in the other, she learns that certain collections count as pairs, or fours, or sevens. And this is a first step toward grasping mathematical structures:

The process... may not go all the way to characters and strings as completely freestanding abstract objects, but the development goes pretty far in that direction. Presumably, nothing philosophically occult or scientifically respectable has been invoked along the way. In the end, we either demystify numbers [and abstract structures] or we mystify more mundane items [such as letters of the alphabet]. \cite{Shapiro1997}

In effect, the structuralist argues that in order to explain pattern recognition of the sort employed by young children, we should acknowledge the existence of structures, including the alphabet and the number seven.

Having taken that last step, we appear to face a dilemma: either we insist on thinking of patterns as coextensive with (but not identical to) the elements that comprise them, or we think of them as freestanding and abstract. On reflection, the first option quickly leads to absurd conclusions. For instance, if all tokens of the letter ‘L’ were destroyed and if the letter truly were coextensive with its token instantiations, then the letter itself would perish. Likewise, if no physical collection of some particular cardinality happened to exist at a given moment, the natural number corresponding to that cardinality would itself (perhaps temporarily?) cease to exist. Recall that we already have reason to believe that there are some infinite numbers that are \textit{never} instantiated in concrete collections. Construing patterns as coextensive with the systems they organize is therefore unacceptable. And so, the structuralist invites us to accept the existence of freestanding, abstract patterns.

\textbf{Linguistic abstraction.} Pattern recognition has serious limitations as a means of acquiring mathematical knowledge. In order to grasp a structure in this manner, one must perceive a concrete system which exemplifies it. The maximum numerosity of systems human beings are capable of perceptually distinguishing is an open question.\textsuperscript{14} Shapiro suggests however that it certainly does not exceed ten thousand:

At some point, still early in our child’s education, she develops an ability to understand cardinal and ordinal structures beyond those that she can recognize all at once via pattern recognition and beyond those that she has actually counted or could count. What of the 9422 pattern...? Surely, we do not learn about and teach such patterns by simple abstraction and ostensive definition. The parent does not say, “Look over there, that is 9422” \cite[p.117]{Shapiro1997}

\textsuperscript{14}This is a question I return to in Chapter 4.
The suggestion is that another, more sophisticated process must be invoked to explain our knowledge of more complex structures. Here, language plays an important facilitating role. As I have noted, we begin to understand the meanings of numerals and number-names once we grasp the connection each of these has with appropriately sized collections. At first, this knowledge is gained piecemeal. But having grasped the connection in the case of small collections, we are ready to take the next step: that is, to realize that numerosity patterns themselves form a system. The distinct and systematic labels that language makes available help the child grasp each number as itself an object, rather than as a property of collections. Language moreover helps the child understand that the system of numbers (now construed as objects) itself displays a regular, orderly pattern—a pattern with a further, higher-order property: each of its elements has an unique successor, such that no two elements share a successor. Once the child has grasped this, she has (implicitly) grasped the Peano axioms.

Two pieces of evidence speak in favour of this reconstruction. At a certain moment, children delight in making up labels for absurdly large and sometimes nonsensical numbers (“a billion trillion zillion”) and gleefully naming the next higher cardinal. What they seem to be enjoying is their new-found grasp of structures whose corresponding concrete collections they could not possibly envisage. It’s interesting to note moreover that chimpanzees too can be taught to match labels (including arabic numerals) to collections of items with the appropriate numerosity. Interestingly however, chimps take a roughly equal period of time to learn each label for collections from 0 through 9 [Kawai and Matsuzawa 2000]. Unlike human children, they never seem to ‘get’ that every label must be followed by a successor. One can reasonably hypothesize that our capacity for learning arithmetic somehow piggybacks on our grasp of natural language [Hauser et al. 2002].

**Implicit definition.** Once learners have at their disposal the full semantic resources made available by natural language, it’s possible to communicate the nature of a structure by indirect description. In the case of structures that make no reference to entities outside of themselves, we can describe the elements that comprise them strictly in terms of the relationships they bear to one another. The system thus described need not have been observed or even to have physically existed. We begin by holding true a plausible collection of propositions or axioms in which some undefined term $T$ appears. Perhaps the collection strikes us as self-evident. The term $T$ comes to be implicitly (or functionally) defined by the axioms; it comes to possess whatever meaning it needs to for the statements to come out true. Shapiro suggests that the strategy succeeds provided that two conditions

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15There is a second, independent reason why pattern recognition cannot be the whole story: pattern recognition ties mathematical knowledge to sensory experience, while mathematics is typically held up as an example of the *a priori*. 


are met. The first is that the sentences that serve to specify $T$ must be consistent; that is, they must be capable of being true simultaneously. Second, the structure specified by the axioms must be unique. Any systems that the definitions hold good of must share a common structure.\footnote{For a somewhat critical discussion of the powers of implicit definition, see Horwich [1997].}

### 2.2.2 Counter-evidence

Shapiro’s\footnote{Shapiro [1997]} account of mathematical knowledge is admirably clear and precise in its commitments. Thanks to its clarity, it is possible to critically examine some of its details. I will start by reevaluating whether linguistic abstraction and implicit definition can indeed account for human knowledge of large structures. I will return to pattern recognition and smaller, finite structures a little later on.

**Large structures.** According to Shapiro, human beings capable of recognizing relatively small patterns are bootstrapped into an understanding of large (and sometimes truly vast) structures by linguistic abstraction and implicit definition. It’s worth underscoring that both of these abilities rely crucially on natural language. The theory is quite explicit on this point:

> [These] epistemic techniques suggest a tight link between grasp of language and knowledge of structures. This is especially true for implicit definition. For the fields of pure mathematics at least, grasping a structure and understanding the language of its theory amount to the same thing. There is no more to understanding a structure and having the ability to refer to its places than having an ability to use the language correctly…

> [T]he way humans apprehend structures and the way we “divide” the mathematical universe into structures, systems, and objects depends on our linguistic resources. Through successful language use, we structure the objective subject matter. Thus, language provides our epistemic access to mathematical structures. [Shapiro 1997, p.137]

Let me concede for the moment that natural language could, in principle, play the required mediating role between acausal structures and our cognitive apparatus. This is a point to which I will return. For now, I want to focus on some of the implications of the proposal. If Shapiro is right—if our epistemic access to mathematical facts is mediated by language—then we would expect severe linguistic impairments to have a deleterious impact on our mathematical abilities. And conversely, we might perhaps also expect impairments of mathematical intelligence to correlate to a degree with impaired linguistic ability (though this is far less certain). A look at the psychological literature does not bear out either of these predictions.
Semantic dementia. The first piece of evidence I’d like to look at comes from studies of dementia. In general, dementias are characterized by a chronic decline in cognitive function across two or more distinct domains. They normally onset gradually and their early symptoms can be relatively mild. The precise clinical profile of affected patients is hard to predict in advance due to the considerable range of underlying neuropathologies. Still, over time dementias are typically more debilitating than those impairments—such as aphasias, ataxias or amnesias—which target only one type of function (so language, coordination, and episodic memory, respectively) [Albert et al. 1999].

My interest here is in semantic dementia. The condition involves a gradual degeneration of semantic memory, typically due to the atrophy of the cerebral left temporal lobe and supporting tissue (Figure 3). Semantic dementia leads to a loss of understanding of the meanings of both spoken and written words, as well as severe difficulties in articulating content. It also results in an inability to recognize objects, faces and pictures. Since other cerebral regions are typically spared, these deficits are circumscribed and most other aspects of mental life remain unaffected. Patients are typically alert and orient normally in their surroundings. Their perceptual faculties, autobiographical (episodic) memory, and problem solving skills remain intact. In fact, patients can even retain the non-semantic (so syntactic and phonological) aspects of their linguistic competence.

Cappelletti et al. [2001] investigate the extent to which a patient affected by semantic dementia retains an understanding of specifically arithmetic concepts and operations. The subject, IH—a 65-year-old, male, right-handed, British banker—was first diagnosed in 1995. IH’s initial symptoms included severe difficulties with finding words and also with naming objects. IH remained fluent and his speech was syntactically correct but the investigators characterized his replies as ‘vague’ and ‘discursive’, often lacking a clear meaning. To compensate, IH frequently relied on set phrases such as “I delved into that…” or “I am totally committed to…” He was also incapable of reading newspapers. It’s important to emphasize however that IH’s semantic difficulties were not due to a lack of general intelligence. His episodic memory was largely spared and he continued to recognize people and places. His knowledge of familiar topics, such as sports and politics, also remained intact. The problems IH was experiencing seemed therefore almost wholly connected to his knowledge of language.

17 This has sometimes also been called progressive aphasia.
There was one notable exception, however: several years earlier IH had begun to display a lack of judgement while gambling. This ultimately led to financial difficulties and divorce. It’s not clear whether these symptoms were due to temporal lobe atrophy or perhaps to damage to underlying, subcortical structures.

The initial investigation comprised a battery of tests designed to measure IH’s non-mathematical, semantic knowledge. The stimuli included images of both living and human-made entities drawn from six categories: vegetables, body parts, and animals, as well as furniture, modes of transportation, and musical instruments. In each case, the stimuli ranged from the very typical (tomato, dog) to the atypical (asparagus, zebra). Both verbal and pictorial tests were carried out. The verbal tests asked IH to perform such tasks as picture naming, picture matching by type, producing words semantically associated with a given stimulus, offering verbal definitions of objects, and naming as many objects of a particular type as possible. The results were almost uniformly discouraging. IH was wholly unable to perform these tasks, scoring zero on all but name-to-picture matching. There, since the response options were limited, he performed at chance. Age-matched control subjects, by contrast, performed at 97% or better on all tasks. We must therefore conclude that IH’s verbal semantic knowledge was all but nonexistent.

The principal purpose of the study was, of course, to investigate IH’s specifically mathematical abilities. If these abilities depend on our capacity to grasp linguistic meaning, one would expect IH’s understanding of mathematical concepts and operations to be very poor. If, on the other hand, the two are importantly distinct, at least some mathematical understanding should be spared in spite of IH’s other semantic difficulties. Two broad types of tasks were used: the first focused on comprehension and transcoding, the second on calculation. In the first set of tests, IH performed nearly flawlessly. The tasks included recognizing written numbers, counting, naming a number’s successor and predecessor, transcoding from

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18 The sole exception to this pattern of results concerned the naming of seasons, days of the week, and months of the year where IH scored 21 out of a possible 23. This, together with the rest of the results, raises the question whether rote patterns such as the list of the days of the week are encoded differently from other semantic memories.

19 In the category fluency test, IH named zero objects of the types used; the controls averaged almost 15 per minute across the categories.

20 IH did somewhat better on pictorial tests of semantic knowledge. Here, the tasks involved understanding and manipulation of pictures chosen so as to parallel stimuli used in the previous tasks. IH was asked to classify pictures both at the entry-level category (animal, furniture) and at the subcategory level (exotic animals). He was also asked to draw size comparisons between depicted items, perform semantic picture-picture associations, and distinguish between real and nonsense objects based on silhouettes. In each case, the responses IH was asked to make were nonverbal to block the interpretation that his deficits had to do with linguistic articulation. He scored 80% on the picture classification at the category level and 66% at the subcategory level. Similarly, he scored 65% on the size-judgement task, and 70% on the silhouette reading task. (Predictably, the controls’ scores were nearly perfect for all but the silhouette task.) The results suggest that while IH’s linguistic semantic abilities were almost wholly compromised, he did retain some ability to understand objects and their properties.
arabic numerals to spoken number words and vice-versa. He made one mistake (in ten trials) when asked to bisect numbers. The only exception to his apparently nearly perfect comprehension of numbers involved knowledge of number facts. He was not able to say how old he was, what his shoe size was, or how many months there were in the year. Moreover, interestingly, he was not able to name or explain the arithmetical the operators. Still, IH clearly retained much of his understanding of numbers and their properties.

Given IH’s apparent difficulties with explaining the arithmetic operators, one might expect him to have trouble with written calculation. Not so. In fact, he scored above 95% on 2 and 3 digit addition and subtraction problems and 69% and 62% on multiplication and division problems, respectively. His single-digit arithmetic performance was even better. It seems therefore that his knowledge of these operations is largely preserved in spite of the clear semantic impairments he displays. (Cf. Table 1, below.)

<table>
<thead>
<tr>
<th>Task</th>
<th>IH % correct</th>
<th>Controls % correct</th>
</tr>
</thead>
<tbody>
<tr>
<td>Oral single-digit arithmetic (N=254)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Addition</td>
<td>98</td>
<td>100</td>
</tr>
<tr>
<td>Subtraction</td>
<td>95</td>
<td>100</td>
</tr>
<tr>
<td>Multiplication</td>
<td>73</td>
<td>90</td>
</tr>
<tr>
<td>Written multidigit arithmetic (N=128)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Addition</td>
<td>99</td>
<td>99</td>
</tr>
<tr>
<td>Subtraction</td>
<td>96</td>
<td>98</td>
</tr>
<tr>
<td>Multiplication</td>
<td>69</td>
<td>95</td>
</tr>
<tr>
<td>Division</td>
<td>62</td>
<td>95</td>
</tr>
<tr>
<td>Approximate calculation</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Approximation to correct result</td>
<td>not understood</td>
<td>100</td>
</tr>
<tr>
<td>Placing numbers on a line (N=100)</td>
<td>100</td>
<td>100</td>
</tr>
</tbody>
</table>

We should conclude therefore that while IH has very serious nonverbal semantic memory deficits, his understanding of numbers and of arithmetic procedures remains largely intact. This strongly suggests that our understanding of mathematical and of linguistic items involve independent cognitive systems. It therefore gives us our first evidence that mathematical understanding operates independently of some aspects of our linguistic competence.

Agrammatic aphasia. The case of IH leaves open the possibility that our mathematical abilities are bound up closely with syntactic (rather than semantic) aspects of our linguistic competence. Evidence showing that this is not the case comes from work on agrammatic aphasia.
In what (to my knowledge) is the first study of its kind, Varley et al. [2005] investigated the arithmetic abilities of three profoundly aphasic men: S.A., S.O., and P.R. All three subjects were in their late 50s. All three had suffered lesions to their left middle cerebral artery resulting in extensive damage to the left perisylvian temporal, parietal and frontal cortices. Consistent with this damage, the subjects displayed severe but circumscribed linguistic deficits. All three performed above 85% on both spoken and written word-picture matching tasks. And two of the three did relatively well (> 75%) on spoken and written synonym matching tasks. In each case, the subjects’ phonological memory was also relatively spared. Nevertheless, the subjects performed poorly on tests of grammatical processing involving the matching of reversible spoken and written sentences to pictures depicting relevant actions (for example “The man killed the lion” and “The lion killed the man”), scoring below chance on this task. Since their word-knowledge and linguistic memory were apparently not a factor, their failure on this task can only be attributed to a specific grammatical deficit.

Interestingly, all three subjects retained considerable mathematical competence. They were able to add, subtract, multiply and divide whole numbers. They were also able to add and subtract fractions. Moreover, in spite of their difficulties with reversible sentences, the subjects did relatively well with reversible subtraction and division problems. The simplest task of this sort involved solving pairs of arithmetic expressions—such as $59 - 13$ and $13 - 59$, $60 \div 12$ and $12 \div 60$. In order to arrive at the answer, the subjects needed to keep track of the order of presentation and understand its impact on the calculation being performed. A second, slightly harder task of a similar nature used bracketed expressions ($36 \div (3 \times 2)$) that the subjects were asked to solve. Once again, the subjects performed relatively well. The third and hardest task had the subjects insert brackets into unbracketed expressions (such as $7 + 4 \times 3 + 17$). Here, the subjects were deemed to have succeeded on a trial if they were able to insert the brackets in two distinct ways into the given expression so as to produce two different answers. The arithmetic task results are summarized in Table 2, below.

---

21 Also relevant is a commentary by Brannon [2005] and the work of Gelman and Butterworth [2005].
22 The remaining subject scored 70% on the written part of this test.
<table>
<thead>
<tr>
<th>Task</th>
<th>S.A</th>
<th>S.O</th>
<th>P.R.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Calculation tests (N=20)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Addition</td>
<td>19</td>
<td>16</td>
<td>20</td>
</tr>
<tr>
<td>Subtraction</td>
<td>19</td>
<td>19</td>
<td>19</td>
</tr>
<tr>
<td>Multiplication</td>
<td>19</td>
<td>13</td>
<td>17</td>
</tr>
<tr>
<td>Division</td>
<td>19</td>
<td>11</td>
<td>16</td>
</tr>
<tr>
<td>Adding and subtracting fractions (N=30)</td>
<td>27</td>
<td>27</td>
<td>20</td>
</tr>
<tr>
<td>Reversibility tests (N=40)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Subtraction</td>
<td>40</td>
<td>35</td>
<td>37</td>
</tr>
<tr>
<td>Division</td>
<td>37</td>
<td>34</td>
<td>38</td>
</tr>
<tr>
<td>Bracket expressions</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Calculation accuracy (N=64)</td>
<td>45</td>
<td>52</td>
<td>43</td>
</tr>
<tr>
<td>Bracket generation and calculation (N=5)</td>
<td>4</td>
<td>4</td>
<td>2</td>
</tr>
</tbody>
</table>

On the basis of the above results, we must conclude that all three test subjects were capable of coping with complex hierarchical structures in the context of arithmetic operations but not of language-problems. It seems that they could understand and apply hierarchical, syntactic reasoning regarding arithmetic problems that they were unable to bring to bear on linguistic expressions. This suggests that mathematical processing can operate independently of the processing of natural language syntax.

**Savants.** Work presented thus far suggests a certain degree of independence between human numerical and linguistic abilities. Nonetheless, it doesn’t yet pose an insurmountable problem for accounts of mathematical knowledge such as Shapiro’s. An ante rem structuralist can maintain that a grasp of natural language (syntax or semantics) is required in order to initially learn mathematical concepts and operations; once these are understood, however, our mathematical abilities operate (or degrade) independently of language.\(^{23}\) My next piece of evidence is intended to block this move. It concerns the highly unusual skills of an autistic savant calculator.

Mathematical savants are able to appreciate relations between numbers not apparent to the rest of us. The mathematician G. H. Hardy (whom we encountered in Chapter 1) reports a conversation with Ramanujan, a mathematical genius, while the latter lay dying of tuberculosis in a British sanatorium.

\(^{23}\) *Ante rem* structuralists are not the only ones who advance this hypothesis. The dependence of arithmetic competence on natural language is also defended by Hauser et al. [2002]; limited empirical support is offered by Donlan et al. [2007].
“The taxi that I hired to come here bore the number 1729,” said Hardy. “It seemed a rather dull number,” “Oh no, Hardy” replied Ramanujan. “It’s a captivating one: It’s the smallest number that can be expressed in two different ways as a sum of two cubes”—$1729 = 1^3 + 12^3 = 10^3 + 9^3$. [Quoted in Dehaene 1997 p.148]

The intuitive familiarity with natural numbers required to make this sort of observation is a rare gift that is far from being understood by modern cognitive science. The ability reportedly correlates (in many cases, at least) with a certain sensibility which facilitates mathematical research. Indeed, a number of gifted mathematicians, including Gauss, were calculating prodigies. Nonetheless, Dehaene [1997] argues that at least some apparently superhuman feats of calculation rely on heuristics that can be learned and practised. Moreover, calculation ability alone does not necessarily correlate with general intelligence or the capacity to construct imaginative solutions to novel abstract problems.

The case that interests me here is that of Michael, a young savant calculator. Michael is doubly unusual: not only is he a gifted calculator, he is also profoundly autistic. Autism has become something of a cause célèbre in the past two decades. The diagnosis spans a range of disorders whose physiological basis is still not well understood. Autism-spectrum disorders can however be characterized cognitively as involving a characteristic pattern of executive, social, and linguistic deficits. Many autists engage in stereotyped, repetitive behaviours and display obsessive interests. They dislike changes in routine. And they have trouble shifting attention in a flexible and appropriate manner. When focused on a stimulus, they display a bias for local, part-oriented processing. Perhaps for this reason, they seem not to succumb to some visual illusions involving gestalt patterns. Three quarters of autists have an IQ in the mentally retarded range, though some are of average or even above average intelligence. Even high-functioning autists have trouble attributing mental states to others (or to themselves) and so they typically have severe difficulties interpreting or predicting others’ behaviour in terms of beliefs or desires. Many autists display serious difficulties with linguistic communication, with special difficulties in the area of pragmatics—they tend to grasp the literal rather than the intended meaning of what is said. They don’t get jokes or respond appropriately to metaphor [Frith and Happé 1999, Tager-Flusberg et al. 2001].

Michael presents with the classic symptoms of autism with respect to executive function, social behaviour and linguistic ability. He does relatively poorly on tests of general intelligence (IQ 67). However, he is fascinated by jigsaw puzzles which he solves equally well with the picture-side up or down. He is also interested in calendars. Thus, on tests of intelligence involving only abstract shapes and relations, Michael scores well above average (IQ 140). Michael never initiates social interaction and shows no interest in it. Nor does he have any interest in communicating with others. He does not point and does not attend to pointing. He was once taught some rudimentary signs but never uses these spontaneously. Finally, Michael has never learned to speak and shows no indication of understanding spoken language. In fact, he is, by all accounts, completely aloguistic. This is, I realize, a
surprising and radical claim, so I quote the relevant descriptions in full:

Michael is a young man without any speech or verbal comprehension. As a young child, he did not talk or attempt to engage in any kind of communication. He still cannot speak but has learned to copy numbers and letters, though only very poorly. [Anderson et al. 1999 p.385-7]

He did not look at things when someone pointed at them, never waved goodbye or responded to cuddling. Michael is not deaf, but he seemed not to understand any language at all and did not himself develop any speech. He has remained entirely without language, and though he was taught some sign language gestures he never used these spontaneously. [Hermelin 2001 p.109]

He never initiated gestures, such as pointing or waving goodbye. He never began to speak and did not respond to language. He took very little interest in adults and did not try to communicate in any way. He began attending a special school for autistic children at age six. He learned to ‘write’ with a pencil, i.e., he learned to copy letters and numbers. But he has not improved in this skill since his schooldays and his written numbers are often difficult to make out. He also learned a few elementary Paget Gorman signs, though he never used them spontaneously. [Hermelin and O’Connor 1990 p.165]

In one case, there is some suggestion that Michael’s linguistic handicap runs even deeper than the above passages suggest. [Anderson et al. 1999] write that not only is Michael a-linguistic; he also lacks the underlying resources to categorize pictures of concrete objects into the relevant categories (such as vegetable, mode of transport, and so on). This appears to be indicative of a profound semantic deficit.

Michael lacks any language production or comprehension. We might go further, and suggest from his performance on the Columbia [inclusion/exclusion test] that he may also be unable to abstract from objects the semantic categories to which they may be assigned. It may be that Michael demonstrates above average intelligence only when problems are limited to spatial and perceptual dimensions. Moreover, it seems clear that his capacity to deal with problems as long as they do not involve a semantic classification of objects in the real world also extends to numbers. [Anderson et al. 1999 p.399]

It would appear then that Michael’s grasp of natural language (syntax, semantics and pragmatics) is essentially nil.

As it happens, both of Michael’s parents are mathematicians. He was taught to ‘read’ and copy numerals quite early on, though (as we just saw) his writing is often hard to decipher. What is perhaps more surprising given the extent of his various deficits is that Michael is capable of performing basic arithmetic operations on numbers. He can add, subtract, divide, and multiply. Moreover, Michael is capable of factoring numbers. This last ability was the focus of Hermelin and O’Connor’s [1990] study.
Three different tasks were used to test the savant’s skills: recognizing and generating primes, as well as factorizing non-primes. Each task was performed with three-digit, four-digit and five-digit numbers, so at three levels of difficulty. In each case, the task was modeled for Michael twice. After this he was able to proceed with most of the tasks “appropriately and without hesitation.” In one case, a trial had to be rerun to secure Michael’s cooperation. Table 3 compares Michael’s performance to that of a control subject, a male psychologist with a degree in mathematics.

<table>
<thead>
<tr>
<th>Task</th>
<th>Control</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Correct</td>
<td>Mean time (sec)</td>
<td>Correct</td>
</tr>
<tr>
<td>Recognizing</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>301-393</td>
<td>20/30</td>
<td>11.46</td>
<td>29/30</td>
</tr>
<tr>
<td>1201-1309</td>
<td>18/30</td>
<td>12.90</td>
<td>22/30</td>
</tr>
<tr>
<td>10307-10427</td>
<td>23/30</td>
<td>10.73</td>
<td>15/30</td>
</tr>
<tr>
<td>Generating</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>227-281</td>
<td>8/10</td>
<td>12.9</td>
<td>9/10</td>
</tr>
<tr>
<td>1019-1091</td>
<td>5/10</td>
<td>25.6</td>
<td>5/10</td>
</tr>
<tr>
<td>10037-10133</td>
<td>4/10</td>
<td>50.0</td>
<td>5/10</td>
</tr>
<tr>
<td>Factorizing</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>212-221</td>
<td>8/10</td>
<td>22.6</td>
<td>9/10</td>
</tr>
<tr>
<td>1001-1011</td>
<td>7/10</td>
<td>25.5</td>
<td>8/10</td>
</tr>
<tr>
<td>10002-10013</td>
<td>4/10</td>
<td>48.0</td>
<td>7/10</td>
</tr>
</tbody>
</table>

*Note: Number of correct responses and mean decision times.*

In general, both subjects’ error rates as well as the kinds of mistakes committed were similar. In general, the control tended toward omission errors while Michael tended to produce false positives. The most striking difference between the data concerns the response times. In general, the speed of information processing as measured by reaction time studies is closely associated with the level of general intelligence [Jensen 1979]. In a prior study of idiot-savant calendrical calculators [Hermelin and O’Connor 1983] had found that the “simple and complex visual RT of these subjects was in accordance with those expected from their IQ whereas their speed of calendrical calculation was much faster than that usually obtained from people with much higher IQs.” This suggests that savant calculators possess a cognitively-specific calculating ability.

It’s hard to deny that Michael possesses genuine mathematical knowledge. This is particularly clear from his ability to generate prime numbers in the range between 10037 and 10133. (Recall that Shapiro [1997] specifically denies one could grasp structures as large as 9422 by simple pattern recognition.) Moreover, we can be confident that his knowledge
is the same in kind as that displayed by the neurotypical control. An analysis of response times suggests that Michael employs Eratosthenes’ sieve, the same algorithm used by the control. The algorithm involves dividing the target number by all prime numbers less than or equal to the target’s square root. Thus, for example, since 59 cannot be divided by 2, 3, 5, or 7, we can safely conclude that it is prime. So while Michael’s access to mathematical knowledge is exceptionally fast, and perhaps unconscious, the knowledge itself appears to be the same in kind as that possessed by the rest of us. It follows that a grasp of language is not strictly required for genuine mathematical knowledge. It would seem therefore that any account of human mathematical knowledge which holds that a knowledge of language is required in order to grasp complex mathematical structures, such as the natural number structure, is empirically disconfirmed.\footnote{This includes Shapiro [1997] but also Maddy [2007], and arguably the fictionalist accounts of Yablo [2001], and Hoffman [1999].}

\textit{Dyscalculia.} In order to demonstrate the mutual independence of two cognitive capacities, it’s important to show a double dissociation: that is, to prove that each can operate (or fail) independently of the other. The last piece of evidence I’d like to present is intended to show that our mathematical competence can fail while leaving the rest of our cognitive capacities intact. Strictly speaking, this is not crucial to the case I am trying to build. But it does serve to reinforce the conclusion that mathematics and language are subserved by functionally independent cognitive systems.\footnote{A discussion of evidence demonstrating a parallel neurophysiological dissociation (e.g., Kadosh et al. [2007]) will have to wait until Chapter 4.}

The evidence here comes from studies of dyscalculia, a relatively common but still not well understood developmental disorder [Butterworth 2005]. Dyscalculia seems to affect at least 3.6% of the population—so roughly as many people as dyslexia. Those affected show a persistent impairment learning and remembering arithmetic facts as well as problems executing calculating procedures. Of course, reasons for poor math skills among children can vary widely. They can include poor teaching, weak study skills, anxiety, missing lessons, and so on. The situation of dyscalculics however is qualitatively different. Their inability to learn is due to persistent problems in representing and retrieving arithmetic information from long-term semantic memory. This manifests as a lack of intuitive knowledge of even the most basic arithmetic. Thus, in spite of normal intelligence, good instruction and concerted effort, dyscalculics can literally fail to understand what a teacher is saying:

\begin{quote}
\textit{Child 5}: Oh, there’s this really hard thing, about when you’re doing times—Ms. S says you can’t take away this number, but I keep on taking it away, I don’t understand one single bit of it.
\textit{Child 2}: I sometimes don’t understand whatever she (the teacher) says.
\textit{Child 1}: I don’t forget it, I don’t even know what she’s saying.
\end{quote}

[Butterworth 2005]
Along with problems with grasping the relevant facts, dyscalculics show impairments executing calculation procedures. When they do add, subtract, multiply or divide, they typically do so much more slowly. Their performance is error-prone and they lack confidence in their results. And, even as adults, they rely on immature strategies, such as finger-counting.

Butterworth [1999] presents an interesting case study of dyscalculia. “Charles” is an intelligent and resourceful university graduate in his thirties. He has a degree in psychology and works as a psychological counsellor. As one might expect, he copes well with daily life. However, Charles has had severe difficulties with mathematics since childhood. He cannot add up the price of groceries, count the money in his wallet, or figure out the correct change. When tested, he proved completely unable to solve two-digit subtraction problems. He cannot work out multiplication problems involving numbers greater than 5. And although he can find the solution to single-digit addition and subtraction problems, his performance on these is four times slower than a control subject’s (so roughly three seconds). Perhaps Butterworth’s most extraordinary findings concern Charles’ performance on tasks thought to involve very low-level cognitive abilities. One of these is simple number comparison. In general, the time taken by math-typical subjects to compare two single-digit numbers is (roughly) inversely proportional to the difference between them; it’s easier to judge that 2 is smaller than 9 than it is to judge that 8 is. In Charles’ case, this pattern is reversed. The time it takes for him to compare two numbers is proportional to the difference between them. This suggests that he is forced to perform number comparison tasks in a way entirely unlike that of typical subjects. This supposition is reinforced by subitizing data. Math-typical subjects take almost the same amount of time to grasp (or ‘subitize’) the numerosity of collections containing one, two and three items. This capacity is thought to be a very low-level cognitive or perhaps even perceptual ability.26 Interestingly, Charles does not subitize; he laboriously counts items even in patterns containing two or three entities. This, once again, reinforces the conclusion that dyscalculics are affected by a pervasive, math-specific cognitive impairment.27

Discussion. We have now seen some of the evidence pointing to a functional double dissociation between human linguistic and mathematical abilities. In Chapter 4, we will review additional evidence showing that the processing of linguistic and mathematical information is subserved by independent cortical regions. I want to emphasize that the emerging picture is far from straightforward. We need to keep in mind that certain linguistic and mathematical (specifically, arithmetic) abilities do appear to interact in certain circumscribed


27Before moving on, I should mention that (unlike Charles) many dyscalculics discussed in the literature present with a variety of additional cognitive deficits. Some display general working memory problems, left-right discrimination difficulties, spatial and psychomotor deficits, agnosia, dysgraphia, and reading problems. Dyscalculia and dyslexia in particular frequently co-occur. Recent work (for example Landerl et al. [2004]) suggests that while such comorbidity is common, it is dissociable from the core math-specific problems.
respects. For instance, Welsh and Chinese-speaking children learn the count sequence faster than French or English speakers. This can be attributed to the fact that number-terms in the former two languages are perfectly regular, whereas French and English display tricky exceptions to a regular pattern (‘quatre-vingt’, ‘eleven’) [Miller et al. 2005]. Moreover, studies from Amazonia by Gordon [2004] seem to suggest that users of languages which lack count terms beyond the first three are impaired in their arithmetic abilities. However, in spite of limited interactions, the examples discussed above show that sophisticated mathematical and linguistic capacities can develop and operate independently. And if that’s right then any theory which claims that mathematical competence piggybacks on linguistic competence is committed to an empirically false picture. We have seen that this is precisely what the ante rem structuralist’s account of our knowledge of large structures maintains. And so that account must be amended or rejected.

Small structures. As we have seen, Shapiro [1997] himself holds that perceptual pattern recognition does not suffice to explain our ability to track facts about complex mathematical structures—those which comprise more than several hundred places:

One cannot grasp a structure $S$ by simple pattern recognition unless one can perceive a system that exemplifies $S$. Such a structure can have at most a small, finite number of places. [p.129]

Still, one could readily imagine an abstract realist adopting a more sanguine stance. Such a theorist might argue—perhaps taking her cue from the cognitive mechanisms discussed by Maddy [1990]—that, when correctly understood, perceptual pattern recognition does indeed afford us epistemic access to even the most arcane mathematical posits. I now want to argue that, on the contrary, pattern recognition (and similar processes) are wholly incapable of offering us an explanation of our knowledge of purported abstract facts.

Let’s begin from some shared assumptions. I take it as given that there are mathematical facts and we have true beliefs about some of them. Of course, we also sometimes make mistakes. But when these are spotted, we respond by changing our minds, tweaking our beliefs, and continuing on with our mathematical research. All this is uncontroversial.

Two aspects of this situation deserve to be distinguished and attended to. The first is aboutness. Our mathematical beliefs are, of course, about mathematical entities. (What

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28Alternatively, an abstract realist might argue that if pattern recognition does not offer us epistemic access to abstracta then some other, relevantly similar procedure does. See, for instance, Resnik [1997]. While I will keep the discussion focused on pattern recognition for the sake of clarity, the overall argumentative strategy employed here extends readily to other such views.

29Or, even if it sounds controversial to the antirealist, it is a description of the situation that a platonist would be happy to endorse.
else?) This is easy to ignore since aboutness comes so cheap. No contact of any kind need exist between us and what we think about. In fact, what we think about need not so much as exist. We are just as happy thinking about Boston as Gotham, orchids as phlogiston.\textsuperscript{30}

The second aspect of the situation that deserves mention is the \textit{responsiveness} of our beliefs to evidence. We say that a belief is responsive to a set of facts if discovering new information concerning those facts is capable of altering that belief. Typically, when all goes well, our beliefs about real, existing entities are responsive to the states of those entities. Thus, Boston-beliefs are responsive to Boston-facts, orchid-beliefs to orchid-facts, and so forth. This is not the end of the story however. Boston beliefs may additionally be responsive to a host of other facts as well: perhaps facts about baseball, or linguistics, or New York. Sorting out fully and precisely \textit{which} facts a given belief happens to be responsive to is difficult and perhaps even impossible. (Luckily we won’t need to do any of that here.) Notice also that the situation is slightly different in the case of beliefs about the non-existent. Our phlogiston-beliefs cannot be responsive to phlogiston-facts; there aren’t any such facts. Instead, phlogiston-beliefs are responsive to a variety of other states of affairs—including those involving oxygen, combustion, wood, charcoal, and so on. Again, we may not be able to demarcate precisely which states of affairs are relevant and which are not. But, once again, we must recognize that a fuzzy boundary is a boundary nonetheless; the difficulty of making a sharp distinction does not detract from the overall point.

Consider now what our beliefs \textit{about} mathematical entities are responsive to. They cannot be responsive to mathematical facts construed along platonist lines. That’s because the process of altering one’s beliefs is a cognitive process. Cognitive processes supervene on neurophysiological events, by which I mean that no change on the cognitive plane occurs without an accompanying neurophysiological change (though not vice-versa). One of the important and remarkable empirical findings of the past century is that all physical processes—including those that take place in living organisms—are fully causally closed. \textit{Every physical effect is fully determined by law by antecedent physical occurrences} \cite{Papineau2001,Spurrett1999}. On the platonist account, mathematical states of affairs are causally inert. No physical process can therefore be altered in response to acausal facts (even if we allow that such facts obtain). Thus, \textit{a fortiori}, no cognitive process can alter in response to abstract mathematical facts. It follows that when we recognize a mistake in our understanding and change our minds so as to have our beliefs accord with mathematical reality, if our beliefs are responsive to something, that something is not the mathematical facts.

What then are mathematical beliefs responsive to? It might be tempting to insist on the noble origin and purity of mathematics by suggesting that mathematical beliefs have

\textsuperscript{30}The contrast here, of course, is with reference which, by most accounts, requires at least that the entity referred to exist (though see \cite{McGinn2000}). I mention reference only to set it aside until Chapter 3. The important point for us not to let go of is that our mathematical beliefs are \textit{about} mathematical structures and entities.
no need of facts— and certainly not of physical facts! On reflection however, this proves incoherent. If beliefs about mathematicalia are not responsive to acausal facts and they are not responsive to physical facts then they are not responsive to facts, full stop. We have agreed however that we possess mathematical knowledge. Beliefs that are not responsive to (any) facts come in two varieties: they can be irrational idées fixes or they can shift utterly randomly. In either case, such beliefs cannot be constitutive of knowledge. Neither the platonist nor anyone else should be driven to characterize our mathematical beliefs in this way.

There’s another option. One might reasonably suppose that only a posteriori beliefs are responsive to a circumscribed set of facts; perhaps beliefs about mathematicalia are unusual in being responsive not to this or that state of affairs, but rather to the totality of facts. This would perhaps help explain why none of our experiences can disconfirm a true mathematical proposition. There is no doubt a grain of truth here somewhere. Nonetheless, it won’t help the platonist. The sum total of causally-potent, physical facts do not add up to an ante rem, acausal state of affairs. Even were we to accept this possibility, therefore, we would still be compelled to conclude that our beliefs about mathematicalia are not responsive to what the platonist tells us mathematics is about: the facts of mathematics.

If we allow that our beliefs about mathematics are responsive to some set of facts, but not to mathematical facts, some odd consequences follow. Imagine, for instance, a scenario whereby we (somehow) come to be excommunicated from the platonic realm. Less figuratively: imagine that our beliefs about mathematics continue to be responsive to whatever physical facts they are currently responsive to but that platonic facts (somehow) effectively vanish. What, if anything, would the impact of such an unparalleled metaphysical catastrophe be? A moment’s reflection suggests that, in fact, we would not so much as notice. The physical universe would continue to run its course. Our brains would continue to function as they do. And the cognitive processes that working mathematicians undergo would exactly mirror the transitions that they currently undergo. Plausibly, even their phenomenology would remain the same. The same articles would be written. The same proofs would be accepted for publication or rejected. Mathematics would continue to be indispensable to the conduct of natural science. In brief, none of us would be any the wiser. It seems then that mathematicalia construed as platonic abstracta are otiose. They are an unnecessary fifth wheel, an empty place-holder, in explanations of the nature and conduct of the mathematical enterprise. I suppose that one can nonetheless maintain that ante rem states of affairs exist. But I’m not sure what the scientific point of doing that might be. Better, I suggest, to conclude that (P.ii)—the hypothesis that mathematical states of affairs are acausal—is false.

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31I explore this important issue in Chapter 5.

My argument rests on two vulnerable assumptions. The abstract realist can reject the hypothesis that mental states supervene on brain-states, opting perhaps for some form of mind-body dualism. And she is free to reject the hypothesis that physics is causally complete. In either case, since both of these propositions are today widely held by philosophers and scientists, the burden is on her to prove her case. Lacking such proof, the mathematical realist ought to give up on \((P.\text{ii})\). And since \((P.\text{i})\) entails \((P.\text{ii})\), the rejection of the latter results (by modus tollens) in the rejection of the former also. That last step is tantamount to a wholesale rejection of an ante rem conception of mathematics.\(^{33}\)

### 2.3 A new direction

The abstract realists’ inability to furnish a satisfactory account of our epistemic access to mathematical facts is sometimes read as giving credibility to ontological anti-realism. Hartry Field [1989], for instance, comments:

Benacerraf’s challenge—or, at least, the challenge which his paper suggests to me—is to provide an account of the mechanisms that explain how our beliefs about [abstract] entities can so well reflect the facts about them. The idea is that if it appears in principle impossible to explain this, then that tends to undermine the belief in mathematical entities, despite whatever reason we might have for believing in them. [Field 1989, original emphasis.]

One part of what Field says is, I think, exactly right. Until the ante rem realist can offer a plausible epistemic account, we should remain skeptical about her overall conception of the nature of mathematics and its posits. Nonetheless, I think the passage takes a step too far. Abstract realism makes weighty (and questionable) commitments beyond those that an ontological realist must strictly make. The epistemic difficulties abstract realists find themselves ensnared in are logically independent of the arguments for NORM presented in the previous chapter. Field would like to reject the latter, but to do so he would need to challenge them directly.\(^{34}\) As things stand, the mathematical realist is free to reject \((P.\text{i-ii})\), to backtrack to NORM and explore alternative possibilities. This is indeed roughly what I propose to do, but not until Chapter 4. Before we can entertain new realist proposals, some of our background assumptions concerning language need to be modified.

\(^{33}\)Two comments here: first, none of this entails a rejection of structuralism. Plausibly then, the bulk of Shapiro’s [1997] work is very much on the right track in key respects. Second, I still owe a response to the apparently persuasive conceptual arguments for platonism presented in Section 2.1.1. I defer that task to Chapter 5.

\(^{34}\)It’s worth pointing out that even if Field’s own nominalization program were to succeed for all of physics, including quantum mechanics, maths’ role as an indispensable catalyst for successful abduction would still require explanation. For a discussion of this point, see Steiner [1998].
References


